COMPLETENESS OF $\alpha_n \cos nx+\beta_n \sin nx$

BY

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In a recent paper [2], we studied the completeness of $A \cos nx+B \sin nx$ where A and B are arbitrary complex numbers. In this paper, we generalize the completeness question and study the completeness of $\alpha_n \cos nx+\beta_n \sin nx$ where the α_n and β_n are independent of x and depend only on n.

We divide the paper into two parts. In part I, we study the completeness on $[0, \pi]$, while in part II we are on $[-a, a]$ where $0<a<\pi$.

I. **Completeness on** $[0, \pi]$. In this part, we make the assumption that, for each n, at least one of α_n and β_n is nonzero. Accordingly, we can normalize and study the completeness of $\lambda_n \cos nx+\sin nx$ or $\cos nx+\lambda_n \sin nx$ where $|\lambda_n| \leqq 1$.

The main results are

THEOREM 1. $\{\lambda_n \cos nx+\sin nx\}_{n=1}^{\infty}$ *is* $L^2[0, \pi]$ *complete if* $|\lambda_n|<1$, $n=1, 2, \ldots$.

THEOREM 2. *There exist* λ_n *such that* $|\lambda_n|<1, n=1, 2, \ldots$ *and* $\{\cos nx+\lambda_n \sin nx\}_{n=1}^{\infty}$ *is* $L^1[0, \pi]$ *incomplete.*

THEOREM 3. $\{\cos nx+\lambda_n \sin nx\}_{n=0}^{\infty}$ *is* $C[0, \pi]$ *complete if* $|\lambda_n|<1$, $n=1, 2, \ldots$.

THEOREM 4. *There exist* λ_n *such that* $|\lambda_n|<1, n=1, 2, \ldots$ *and* $\{\lambda_n \cos nx+\sin nx\}_{n=0}^{\infty}$ *is* $C[0, \pi]$ *incomplete.*

LEMMA A. *Let* $f(x)$ *be in* $L^2[0, \pi]$ *and let* $a_n=\pi^{-1} \int_0^\pi f(x) \cos nx\, dx$ *and* $b_n=\pi^{-1} \int_0^\pi f(x) \sin nx\, dx$. *Then*

$$\frac{1}{2\pi} \int_0^\pi |f(x)|^2\, dx = \frac{1}{2}|a_0|^2+\sum_{n=1}^{\infty}|a_n|^2 = \sum_{n=1}^{\infty}|b_n|^2.$$

Proof. Let $g(x)$ be defined on $[-\pi, \pi]$ by extending $f(x)$ evenly. Let $h(x)$ be defined on $[-\pi, \pi]$ by extending $f(x)$ oddly. Then $g(x)$ and $h(x)$ are both in $L^2[-\pi, \pi]$.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos nx\, dx = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx = 2a_n, \qquad n = 0, 1, 2, \ldots,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin nx\, dx = 0, \qquad n = 0, 1, 2, \ldots,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} h(x) \cos nx\, dx = 0, \qquad n = 0, 1, 2, \ldots,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} h(x) \sin nx\, dx = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx = 2b_n, \qquad n = 1, 2, \ldots.$$

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Then, by using Parseval's identity, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = 4\left[\frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2\right],$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |h(x)|^2\,dx = 4\sum_{n=1}^{\infty}|b_n|^2.$$

However, since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = \frac{2}{\pi}\int_{0}^{\pi} |f(x)|^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |h(x)|^2\,dx$$

we have the desired result.

THEOREM 1. *Let* $|\lambda_n| < 1$ *for* $n = 1, 2, \ldots$. *Then* $\{\lambda_n \cos nx + \sin nx\}_{n=1}^{\infty}$ *is complete in* $L^2[0, \pi]$.

Proof. Assume there exists $f(x) \in L^2[0, \pi]$ such that

$$\int_0^{\pi} f(x)(\lambda_n \cos nx + \sin nx)\,dx = 0, \qquad n = 1, 2, \ldots. \tag{1.1}$$

Let a_n and b_n be as in Lemma A. Then, (1.1) becomes $\lambda_n a_n + b_n = 0$, $n = 1, 2, \ldots$,

$$\frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2 = \sum_{n=1}^{\infty}|b_n|^2 = \sum_{n=1}^{\infty}|\lambda_n a_n|^2 < \sum_{n=1}^{\infty}|a_n|^2$$

which is impossible unless $a_n = 0$, $n = 0, 1, 2, \ldots$. Since

$$\frac{1}{2\pi}\int_0^{\pi} |f(x)|^2\,dx = \frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2 = 0,$$

we have that $f(x) = 0$ a.e. and have established completeness.

THEOREM 2. *There exist* λ_n *such that* $|\lambda_n| < 1$ *for* $n = 1, 2, \ldots$ *and*

$$\{\cos nx + \lambda_n \sin nx\}_{n=1}^{\infty}$$

is incomplete in $L^1[0, \pi]$.

Proof. Let λ_n be defined as

$$\lambda_n = -\frac{\int_0^{\pi} x \cos nx\,dx}{\int_0^{\pi} x \sin nx\,dx}, \qquad n = 1, 2, \ldots.$$

Integrating by parts, we get that

$$\lambda_n = -\frac{(-1)^n - 1}{n^2} \Big/ \frac{-(-1)^n \pi}{n} = \frac{1-(-1)^n}{n\pi}.$$

Since $(-1)^n\pi/n$ is never zero, our definition of λ_n makes sense. Also, we have $|\lambda_n| = |(1-(-1)^n)/n\pi| \leqq 2/n\pi < 1$. By the way λ_n was defined,

$$\int_0^{\pi} x\,(\cos nx + \lambda_n \sin nx)\,dx = 0, \qquad n = 1, 2, \ldots. \tag{1.2}$$

Since $f(x)=x$ is in $L^\infty[0, \pi]$, (1.2) tells us that $\{\cos nx+\lambda_n \sin nx\}_{n=1}^\infty$ is incomplete in $L^1[0, \pi]$.

LEMMA B. *If* $\{-\lambda_n \cos nx+\sin nx\}_{n=1}^\infty$ *is complete in* $L^2[0, \pi]$, *then*

$$\{\cos nx+\lambda_n \sin nx\}_{n=0}^\infty$$

is complete in $C[0, \pi]$.

Proof. Choose any $\mu(x)$ of bounded variation on $[0, \pi]$, normalized so that $\mu(0)=0$ and $\mu(x)\equiv(\mu(x_-)+\mu(x_+))/2$. Suppose that

$$\int_0^\pi d\mu(x) = 0 \tag{1.3}$$

and

$$\int_0^\pi (\cos nx+\lambda_n \sin nx)\, d\mu(x) = 0, \qquad n = 1, 2, \ldots. \tag{1.4}$$

From (1.3) we get $\mu(0)=\mu(\pi)=0$. Using this fact together with integration by parts in (1.4), gives us

$$\int_0^\pi (-\lambda_n \cos nx+\sin nx)\mu(x)\, dx = 0, \qquad n = 1, 2, \ldots.$$

Since $\{-\lambda_n \cos nx+\sin nx\}_{n=1}^\infty$ is assumed $L^2[0, \pi]$ complete, and since $\mu(x)$, as a bounded function, is in $L^2[0, \pi]$, we must have $\mu(x)\equiv 0$ and have established completeness.

THEOREM 3. *Let* $|\lambda_n|<1$ *for* $n=1, 2, \ldots$. *Then* $\{\cos nx+\lambda_n \sin nx\}_{n=0}^\infty$ *is complete in* $C[0, \pi]$.

Proof. By Theorem 1, $\{-\lambda_n \cos nx+\sin nx\}_{n=1}^\infty$ is complete in $L^2[0, \pi]$. By Lemma B, $\{\cos nx+\lambda_n \sin nx\}_{n=0}^\infty$ is complete in $C[0, \pi]$.

Comparing Theorem 2 with Theorem 3, we see that just the addition of the constant term, can change an $L^1[0, \pi]$ incomplete sequence into a $C[0, \pi]$ complete sequence. One might be tempted to conjecture that since $\{\lambda_n \cos nx+\sin nx\}_{n=1}^\infty$ is $L^2[0, \pi]$ complete if $|\lambda_n|<1$, then $\{\lambda_n \cos nx+\sin nx\}_{n=0}^\infty$ is $C[0, \pi]$ complete if $|\lambda_n|<1$. This is not the case, as the following theorem demonstrates.

THEOREM 4. *Let* λ_n *be periodic with period* 6, *let* $\lambda_5=-\lambda_1$, $\lambda_2=-\lambda_4$ *and* $\lambda_1\lambda_2=-1/3$. *Then* $\{\lambda_n \cos nx+\sin nx\}_{n=0}^\infty$ *is incomplete in* $C[0, \pi]$.

NOTE. In this theorem, we can choose λ_n to satisfy $|\lambda_n|<1$, $n=1, 2, \ldots$.

Proof. Let $g_n(x)=\lambda_n \cos nx+\sin nx$. We will prove that we can find nontrivial C_1, C_2, C_3 and C_4 such that

$$C_1 g_n(0)+C_2 g_n(\pi/3)+C_3 g_n(2\pi/3)+C_4 g_n(\pi) = 0, \qquad n = 0, 1, \ldots. \tag{1.5}$$

Once this is done we will have proven incompleteness since any linear combination of $\{g_n(x)\}_{n=0}^{\infty}$ will also have this property and hence we would be unable to approximate a function $f(x)$ such that

$$C_1f(0)+C_2f(\pi/3)+C_3f(2\pi/3)+C_4f(\pi) \neq 0.$$

Since, obviously there are functions $f(x) \in C[0, \pi]$ with this property we will have proven incompleteness. In finding C_1, C_2, C_3 and C_4 we notice that since λ_n has period 6, $g_n(0)$, $g_n(\pi/3)$, $g_n(2\pi/3)$ and $g_n(\pi)$ all have period 6. Therefore, equation (1.5) has to be satisfied only for $n=0, 1, \ldots, 5$ and all other values of n follow by periodicity. We therefore have the following 6 simultaneous equations to be satisfied nontrivially.

$$(1.6) \qquad n = 0 \quad C_1\lambda_0+C_2\lambda_0+C_3\lambda_0+C_4\lambda_0 = 0,$$

$$(1.7) \qquad n = 1 \quad C_1\lambda_1+C_2\left(\frac{\lambda_1}{2}+\frac{\sqrt{3}}{2}\right)+C_3\left(\frac{-\lambda_1}{2}+\frac{\sqrt{3}}{2}\right)+C_4(-\lambda_1) = 0,$$

$$(1.8) \qquad n = 2 \quad C_1\lambda_2+C_2\left(\frac{-\lambda_2}{2}+\frac{\sqrt{3}}{2}\right)+C_3\left(\frac{-\lambda_2}{2}-\frac{\sqrt{3}}{2}\right)+C_4\lambda_2 = 0,$$

$$(1.9) \qquad n = 3 \quad C_1\lambda_3+C_2(-\lambda_3)+C_3(\lambda_3)+C_4(-\lambda_3) = 0,$$

$$(1.10) \qquad n = 4 \quad C_1\lambda_4+C_2\left(\frac{-\lambda_4}{2}+\frac{\sqrt{3}}{2}\right)+C_3\left(\frac{-\lambda_4}{2}+\frac{\sqrt{3}}{2}\right)+C_4(\lambda_4) = 0,$$

$$(1.11) \qquad n = 5 \quad C_1\lambda_5+C_2\left(\frac{\lambda_5}{2}-\frac{\sqrt{3}}{2}\right)+C_3\left(\frac{-\lambda_5}{2}-\frac{\sqrt{3}}{2}\right)+C_4(-\lambda_5) = 0.$$

By the conditions $\lambda_1=-\lambda_5$ and $\lambda_2=-\lambda_4$ equations (1.10) and (1.11) are the same as equations (1.8) and (1.7) respectively. If we set $C_1=-C_3$ and $C_2=-C_4$ then (1.6) and (1.9) are satisfied. Hence we are just left with (1.7) and (1.8). After we substitute for C_3 and C_4, these two equations become

$$(1.12) \qquad C_1\left(\frac{3\lambda_1}{2}-\frac{\sqrt{3}}{2}\right)+C_2\left(\frac{3\lambda_1}{2}+\frac{\sqrt{3}}{2}\right) = 0,$$

$$(1.13) \qquad C_1\left(\frac{3\lambda_2}{2}+\frac{\sqrt{3}}{2}\right)+C_2\left(-\frac{3\lambda_2}{2}+\frac{\sqrt{3}}{2}\right) = 0.$$

Equations (1.12) and (1.13) can be solved nontrivially iff

$$\begin{vmatrix} \frac{3\lambda_1}{2}-\frac{\sqrt{3}}{2} & \frac{3\lambda_1}{2}+\frac{\sqrt{3}}{2} \\ \frac{3\lambda_2}{2}+\frac{\sqrt{3}}{2} & -\frac{3\lambda_2}{2}+\frac{\sqrt{3}}{2} \end{vmatrix} = 0.$$

This reduces to $\lambda_1\lambda_2+1/3=0$ which is given.

In Theorem 3 we saw that if $|\lambda_n|<1$ for $n=1, 2, \ldots$, then $\{\cos nx+\lambda_n \sin nx\}_{n=0}^{\infty}$ is $C[0, \pi]$ complete. As a proof of the delicateness of that theorem we have the following theorem.

THEOREM 5. *There exists* $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $|\lambda_n|<1$ *for* $n=2, 3, \ldots$ *and* $\{\cos nx+\lambda_n \sin nx\}_{n=0}^{\infty}$ *is incomplete in* $L^1[0, \pi]$.

Proof. Define λ_n as

$$\begin{aligned}\lambda_n &= -\frac{\int_0^\pi (3x^2-\pi^2)\cos nx\,dx}{\int_0^\pi (3x^2-\pi^2)\sin nx\,dx}, \qquad n = 1, 2, \ldots, \\ &= 2/\pi n \qquad n \text{ even}, \\ &= 6\pi n/(\pi^2 n^2-12) \qquad n \text{ odd}.\end{aligned}$$

It is easily proven that $0<\lambda_n<1$ for $n=2, 3, \ldots$. By the way λ_n was defined,

$$\int_0^\pi (3x^2-\pi^2)(\cos nx+\lambda_n \sin nx)\,dx = 0, \qquad n = 0, 1, \ldots.$$

Since $3x^2-\pi^2 \in L^\infty[0, \pi]$, we have the desired incompleteness.

THEOREM 6. *For each n, let* $\lambda_n=1$ *or* -1. *Then* $\{\cos nx+\lambda_n \sin nx\}_{n=0}^{\infty}$ *is complete in* $C[0, \pi]$.

Proof. Assume there exists a finite measure $d\mu_1(x)$ on $[0, \pi]$ such that

$$\int_0^\pi (\cos nx+\lambda_n \sin nx)\,d\mu_1(x) = 0, \qquad n = 0, 1, 2, \ldots.$$

Then

$$\left(\int_0^\pi \cos nx\,d\mu_1(x)\right)^2 = \left(\int_0^\pi \sin nx\,d\mu_1(x)\right)^2, \qquad n = 0, 1, 2, \ldots. \tag{1.14}$$

Let $u=\pi/2-x$. Then

$$\begin{aligned}\cos nx+\lambda_n \sin nx &= \cos nu-\lambda_n \sin nu, & n &= 4K, \\ &= \sin nu+\lambda_n \cos nu, & n &= 4K+1, \\ &= -\cos nu+\lambda_n \sin nu, & n &= 4K+2, \\ &= -\sin nu-\lambda_n \cos nu, & n &= 4K+3.\end{aligned}$$

Let $d\mu(u)=d\mu_1(\pi/2-u)$. Then, by letting $u=\pi/2-x$, (1.14) becomes

$$\left(\int_{-\pi/2}^{\pi/2} \cos nu\,d\mu(u)\right)^2 = \left(\int_{-\pi/2}^{\pi/2} \sin nu\,d\mu(u)\right)^2, \qquad n = 0, 1, 2, \ldots. \tag{1.15}$$

Actually (1.15) holds for $-n$ also. Now let

$$F(z) = \left(\int_{-\pi/2}^{\pi/2} \cos zu\,d\mu(u)\right)^2 - \left(\int_{-\pi/2}^{\pi/2} \sin zu\,d\mu(u)\right)^2.$$

$F(z)$ is an entire function, $F(\pm n)=0$ and $|F(z)| \leqq Me^{\pi|z|}$ for some M. By [1, p. 156], $F(z)\equiv C \sin \pi z$, where C is some constant. However, $F(z)$ is an even function of z while $C \sin \pi z$ is odd. Hence C must be 0. Therefore

$$\left(\int_{-\pi/2}^{\pi/2} \cos zu\,d\mu(u)\right)^2 \equiv \left(\int_{-\pi/2}^{\pi/2} \sin zu\,d\mu(u)\right)^2.$$

Since both

$$\int_{-\pi/2}^{\pi/2} \cos zu \, d\mu(u) \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} \sin zu \, d\mu(u)$$

are entire functions, we must have that

$$\int_{-\pi/2}^{\pi/2} \cos zu \, d\mu(u) = \pm \int_{-\pi/2}^{\pi/2} \sin zu \, d\mu(u),$$

where we have either $+$ for all z or $-$ for all z. In either case we have an even function equal an odd function which is impossible unless both are identically zero.

Therefore

$$\int_{-\pi/2}^{\pi\ 2} \cos nu \, d\mu(u) = \int_{-\pi/2}^{\pi/2} \sin nu \, d\mu(u) = 0, \qquad n = 0, 1, \ldots.$$

Since $\{\cos nx, \sin nx\}_{n=0}^{\infty}$ is complete in $C[-\pi/2, \pi/2]$, $d\mu(u) \equiv 0$. Hence $d\mu_1(x) \equiv 0$ and we have completeness.

THEOREM 7. *Let* λ_n *be real and such that* $|\lambda_n| \leqq 1$, $n = 1, 2, \ldots$. *Then* $\{\lambda_n \cos nx + \sin nx\}_{n=1}^{\infty}$ *is complete in* $L^2[0, \pi]$.

Proof. Assume there exists an $f(x) \in L^2[0, \pi]$ such that

$$\int_0^{\pi} f(x)(\lambda_n \cos nx + \sin nx) \, dx = 0, \qquad n = 1, 2, \ldots.$$

Let a_n and b_n be as in Lemma A. Then $\lambda_n a_n + b_n = 0$, $n = 1, 2, \ldots$,

$$\tfrac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} |\lambda_n a_n|^2.$$

Let $S = \{n : |\lambda_n| < 1\}$ and let $\bar{S} = \{n : |\lambda_n| = 1\}$. Then

$$\tfrac{1}{2}|a_0|^2 + \sum_{n \in S} |a_n|^2 + \sum_{n \in \bar{S}} |a_n|^2 = \sum_{n \in S} |\lambda_n a_n|^2 + \sum_{n \in \bar{S}} |\lambda_n a_n|^2$$
$$= \sum_{n \in S} |\lambda_n a_n|^2 + \sum_{n \in \bar{S}} |a_n|^2.$$

Therefore

$$\tfrac{1}{2}|a_0|^2 + \sum_{n \in S} |a_n|^2 = \sum_{n \in S} |a_n \lambda_n|^2 < \sum_{n \in S} |a_n|^2.$$

This is impossible unless $a_0 = 0$ and $n \in S$ implies $a_n = 0$. Since $\lambda_n a_n + b_n = 0$ we have $b_n = 0$ for $n \in S$. Thus,

$$\int_0^{\pi} f(x)(\cos nx + \gamma_n \sin nx) \, dx = \pi(a_n + \gamma_n b_n)$$
$$= 0 \quad \text{for } n \in S,\ n = 0 \text{ and for any } \gamma_n.$$

In particular let

$$\begin{aligned} \gamma_n &= 1, \qquad & n \in S, \\ &= 1/\lambda_n, \qquad & n \in \bar{S}. \end{aligned}$$

Since λ_n is real and $|\lambda_n| = 1$ for $n \in \bar{S}$ we have that $\gamma_n = +1$ or -1 for any n. By the way γ_n was defined

$$\int_0^\pi f(x)(\cos nx + \gamma_n \sin nx)\, dx = 0, \qquad n = 0, 1, 2, \ldots.$$

Since in Theorem 6 we proved that $\{\cos nx + \gamma_n \sin nx\}_{n=1}^\infty$ is complete in $C[0, \pi]$, it is certainly complete in $L^2[0, \pi]$. Hence $f(x) = 0$ a.e.

COROLLARY. *Let λ_n be real and such that $|\lambda_n| \geqq 1$. Then $\{\cos nx + \lambda_n \sin nx\}_{n=1}^\infty$ is complete in $L^2[0, \pi]$.*

NOTE. In this theorem and in the following one, we do not need all the λ_n to be real. All that is needed is that all those λ_n such that $|\lambda_n| = 1$ should be real.

THEOREM 8. *Let λ_n be real and such that $|\lambda_n| \leqq 1$. Then $\{\cos nx + \lambda_n \sin nx\}_{n=0}^\infty$ is complete in $C[0, \pi]$.*

Proof. By Theorem 7 $\{-\lambda_n \cos nx + \sin nx\}_{n=0}^\infty$ is complete in $L^2[0, \pi]$. By Lemma B, $\{\cos nx + \lambda_n \sin nx\}_{n=0}^\infty$ is complete in $C[0, \pi]$.

THEOREM 9. $\{1, n \cos nx + \lambda \sin nx\}_{n=1}^\infty$ *is complete in* $C[0, \pi]$, *if* $\lambda \neq 2ki$, *k a nonzero integer.*

Proof. If $\lambda = 0$ it is trivially true. Therefore, we can assume $\lambda \neq 0$. Take any function $f(x) \in C[0, \pi]$. We want to approximate by something of the form $\sum_{n=1}^N a_n(n \cos nx + \lambda \sin nx) + a_0\lambda$. Let $P(x) = \sum_{n=1}^N a_n \sin nx + a_0$. Then we want to approximate by $P'(x) + \lambda P(x)$. The idea of the proof is to solve the differential equation $Y' + \lambda Y = f$ and approximate the solution by polynomials. The general solution of the differential equation is $Y(x) = Ce^{-\lambda x} + e^{-\lambda x} \int_0^x e^{\lambda t} f(t)\, dt$ where C is some arbitrary constant. If $\int_0^\pi Y'(x)\, dx = 0$ then $Y'(x)$ can be uniformly approximated by linear combinations of $\{\cos nx\}_{n=1}^\infty$ i.e. given $\varepsilon > 0$ there exist $a_1, \ldots, a_N$ such that

$$\left| Y'(x) - \sum_{n=1}^N a_n n \cos nx \right| < \varepsilon \quad \text{for all } x \in [0, \pi],$$

$$\begin{aligned} \left| Y(x) - Y(0) - \sum_{n=1}^N a_n \sin nx \right| &= \left| \int_0^x \left(Y'(t) - \sum_{n=1}^N a_n n \cos nt \right) dt \right| \\ &\leqq \int_0^x \left| Y'(t) - \sum_{n=1}^N a_n n \cos nt \right| dt \\ &< \varepsilon x \leqq \varepsilon\pi. \end{aligned}$$

Let $a_0 = Y(0)$. Then, since $Y'(x)+\lambda Y(x)=f(x)$

$$\left| f(x) - \sum_{n=1}^{N} a_n(n \cos nx + \lambda \sin nx) - \lambda a_0 \right| < \varepsilon(1+|\lambda|\pi).$$

Since $f(x)$ was arbitrary, we would have that $\{1, n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ is complete in $C[0, \pi]$. All we have to show is that $0=\int_0^\pi Y'(x)\,dx = Y(\pi)-Y(0)$ or that

$$Ce^{-\lambda\pi}+e^{-\lambda\pi}\int_0^\pi e^{\lambda t}f(t)\,dt - C = 0. \tag{1.16}$$

Since $\lambda \neq 2ki$, $e^{-\lambda\pi}\neq 1$. Therefore, we can find C to satisfy (1.16).

THEOREM 10. *$\{n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$, is incomplete in $L^1[0, \pi]$. Moreover, if $\lambda = 2ki$, k a nonzero integer, then $\{1, n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ is incomplete in $L^1[0, \pi]$.*

Proof. $\int_0^\pi e^{\lambda x}(n \cos nx + \lambda \sin nx)\,dx = 0$, $n=1, 2, \ldots$ (and if $\lambda = 2ki$, k a nonzero integer, $\int_0^\pi e^{\lambda x}\,dx = 0$ also). Since $e^{\lambda x} \in L^\infty[0, \pi]$, we have proven incompleteness in $L^1[0, \pi]$.

THEOREM 11. *$\{\lambda \cos nx + n \sin nx\}_{n=0}^{\infty}$ is incomplete in $C[0, \pi]$.*

Proof. As in Theorem 9 we consider a differential equation $Y'-\lambda Y = f$ and want to approximate $Y'(x)$ uniformly by $\sum_{n=1}^{N} a_n \sin nx$. However, this can only be done if $Y'(0)=Y'(\pi)=0$. The general solution of the differential equation is

$$Y(x) = Ce^{\lambda x}+e^{\lambda x}\int_0^x e^{-\lambda t}f(t)\,dt.$$

Hence, $0 = Y'(0) = \lambda C + f(0)$ and

$$0 = Y'(\pi) = \lambda C e^{\lambda\pi}\int_0^\pi e^{-\lambda t}f(t)\,dt + f(\pi).$$

Combining the two equations to eliminate C, we get

$$-f(0)e^{\lambda\pi}+\lambda e^{\lambda\pi}\int_0^\pi e^{-\lambda t}f(t)\,dt + f(\pi) = 0.$$

This suggests an orthogonal measure. Let

$$d\mu(x) = -\delta(x)e^{\lambda\pi}+\lambda e^{\lambda\pi}e^{-\lambda x}+\delta(x-\pi),$$

where $\delta(x)$ is the usual delta measure. An easy calculation shows that this $d\mu(x)$ works i.e.

$$\int_0^\pi (\lambda \cos nx + n \sin nx)\,d\mu(x) = 0, \qquad n = 0, 1, 2, \ldots.$$

THEOREM 12. *$\{\lambda \cos nx + n \sin nx\}_{n=0}^{\infty}$ is complete in $L^P[0, \pi]$ for any $P \geqq 1$.*

Proof. If $\lambda = 0$ it is trivial as we have $\{\sin nx\}_{n=1}^{\infty}$. We can assume $\lambda \neq 0$. Take any $f(x) \in L^P[0, \pi]$. Let $g(x) = Ce^{\lambda x}+e^{\lambda x}\int_0^x e^{-\lambda t}f(t)\,dt$ where C is a constant. Then,

$g(x)$ is absolutely continuous on $[0, \pi]$, differentiable a.e. on $[0, \pi]$ and $g'(x) = \lambda g(x)+f(x)$ a.e. Also, $g'(x) \in L^P[0, \pi]$. As $\{\sin nx\}_{n=1}^{\infty}$ is complete in $L^P[0, \pi]$, there exist $a_1, \ldots, a_N$ such that

$$\left\| g'(x) - \sum_{n=1}^{N} a_n n \sin nx \right\| < \varepsilon \tag{1.17}$$

where $\| \quad \|$ is the $L^P[0, \pi]$ norm. Since $g(x)$ is absolutely continuous $\int_0^x g'(t)\, dt = g(x)-g(0)$. Therefore,

$$\begin{aligned}\left| g(x)-g(0)+\sum_{n=1}^{N} a_n \cos nx - \sum_{n=1}^{N} a_n \right| &= \left| \int_0^x \left(g'(t) - \sum_{n=1}^{N} a_n n \sin nt \right) \right| dt \\ &\leqq \int_0^x \left| g'(t) - \sum_{n=1}^{N} a_n n \sin nt \right| dt \\ &\leqq \int_0^\pi \left| g'(t) - \sum_{n=1}^{N} a_n n \sin nt \right| dt \\ &\leqq \left\| g'(t) - \sum_{n=1}^{N} a_n n \sin nt \right\| \left(\int_0^\pi dt \right)^{(P-1)/P} \\ &< \varepsilon \pi.\end{aligned}$$

Let $a_0 = -g(0) - \sum_{n=1}^{N} a_n$. Then

$$\left| g(x) + \sum_{n=0}^{N} a_n \cos nx \right| < \varepsilon\pi \quad \text{for all } x \in [0, \pi]. \tag{1.18}$$

Therefore

$$\left\| g(x) + \sum_{n=0}^{N} a_n \cos nx \right\| < \varepsilon \pi^{1+(1/P)} \leqq \varepsilon \pi^2.$$

Combining this with inequality (1.17) we get

$$\left\| g'(x) - \lambda g(x) - \sum_{n=0}^{N} a_n (\lambda \cos nx + n \sin nx) \right\| < \varepsilon(1+|\lambda| \pi^2).$$

Since $g'(x) - \lambda g(x) = f(x)$ a.e. we have

$$\left\| f(x) - \sum_{n=0}^{N} a_n (\lambda \cos nx + n \sin nx) \right\| < \varepsilon(1+|\lambda| \pi^2).$$

Since $f(x)$ is an arbitrary $L^P[0, \pi]$ function, we have completeness in $L^P[0, \pi]$.

THEOREM 13. *$\{\lambda \cos nx + n \sin nx\}_{n=1}^{\infty}$ is complete in $L^P[0, \pi]$ for all $P \geqq 1$ iff $\lambda \neq 2Ki$ where K is a nonzero integer.*

Proof. Assume $\lambda \neq 2Ki$. Let $f(x)$ be in $L^P[0, \pi]$.

In Theorem 12 we proved completeness if we are allowed to use the constant function. In inequality (1.18) we had

$$\left| g(x)+\sum_{n=1}^{N} a_n \cos nx + a_0 \right| < \varepsilon\pi \quad \text{for all } x \in [0, \pi],$$

$$\begin{aligned} \left| \int_0^\pi g(x)\,dx + a_0\pi \right| &= \left| \int_0^\pi \left(g(x)+\sum_{n=1}^{N} a_n \cos nx + a_0 \right) dx \right| \\ &\leqq \int_0^\pi \left| g(x)+\sum_{n=1}^{N} a_n \cos nx + a_0 \right| dx \\ &< \varepsilon\pi^2. \end{aligned}$$

Hence if $\int_0^\pi g(x)\,dx=0$ then $|a_0|<\varepsilon\pi$ i.e. we get an arbitrarily small error by forgetting about the constant term. $g(x)=Ce^{\lambda x}+e^{\lambda x}\int_0^x e^{-\lambda t} f(t)\,dt$, and we want to choose C so that $\int_0^\pi g(x)\,dx=0$. If $\lambda=0$, $\int_0^\pi g(x)=C\pi+\int_0^\pi \int_0^x f(t)\,dt\,dx$ and C can obviously be chosen so that $\int_0^\pi g(x)\,dx=0$. If $\lambda\neq 0$ we have

$$\int_0^\pi g(x)\,dx = \frac{C[e^{\lambda\pi}-1]}{\lambda}+\int_0^\pi e^{\lambda x}\int_0^x e^{-\lambda t} f(t)\,dt\,dx.$$

Since $\lambda \neq 2Ki$, $e^{\lambda\pi}-1\neq 0$ and we can choose C so that $\int_0^\pi g(x)\,dx=0$. On the other hand, assume $\lambda=2Ki$ where K is a nonzero integer. Since we want $\int_0^\pi g(x)\,dx=0$ we get

$$\int_0^\pi e^{\lambda x}\int_0^x e^{-\lambda t} f(t)\,dt\,dx = 0.$$

This suggests the bounded linear functional

$$L(f) = \int_0^\pi e^{\lambda x}\int_0^x e^{-\lambda t} f(t)\,dt\,dx$$

$$\begin{aligned} L(\lambda \cos nx + n \sin nx) &= \int_0^\pi e^{\lambda x}\int_0^x e^{-\lambda t}(\lambda \cos nt + n \sin nt)\,dt\,dx, \\ &= \int_0^\pi e^{\lambda x}[-e^{-\lambda x}\cos nx + 1]\,dx \\ &= \int_0^\pi (-\cos nx + e^{\lambda x})\,dx \\ &= \frac{e^{\lambda\pi}-1}{\lambda} = 0 \quad \text{since } \lambda = 2Ki. \end{aligned}$$

Hence $\{\lambda \cos nx + n \sin nx\}_{n=1}^\infty$ is incomplete in $L^p[0, \pi]$.

II. **Completeness on** $[-a, a]$, $0<a<\pi$. In this part we make the assumption that $\alpha_n=P(n)$ and $\beta_n=Q(n)$ where $P(z)$ and $Q(z)$ are polynomials. In addition, we can assume that $P(z)\not\equiv 0$ and $Q(z)\not\equiv 0$, since, in either case, it is well known that we have $L^1[-\varepsilon, \varepsilon]$ incompleteness for any $\varepsilon>0$.

The main results are

THEOREM. (a) *If* $P(z)Q(-z)+P(-z)Q(z)\not\equiv 0$, *then* $\{P(n)\cos nx + Q(n)\sin nx\}_{n=0}^{\infty}$ *is* $C[-a, a]$ *complete if* $0<a<\pi$.

(b) *If* $P(z)Q(-z)+P(-z)Q(z)\equiv 0$, *then* $\{P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ *is* $L^1[-\varepsilon,\varepsilon]$ *incomplete for any* $\varepsilon>0$.

We will use the following theorem [3, p. 186].

THEOREM A. *Let* $F(z)$ *be analytic and of the form* $O(e^{k|z|})$ *where* $K<\pi$, *for* $\operatorname{Re} z\geqq 0$ *and let* $F(z)=0$ *for* $z=0, 1, 2, \ldots$. *Then* $F(z)=0$.

THEOREM 14. *Let* $D(z)=P(z)Q(-z)+P(-z)Q(z)$ *and let a be such that* $0<a<\pi$. *Then, if* $D(z)\not\equiv 0$, $\{P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ *is complete in* $C[-a, a]$.

Proof. Assume there exists a measure $d\mu(x)$ such that

$$P(n)\int_{-a}^{a}\cos nx\, d\mu(x)+Q(n)\int_{-a}^{a}\sin nx\, d\mu(x) = 0, \qquad n = 0, 1, 2, \ldots.$$

Let $F(z)=P(z)\int_{-a}^{a}\cos zx\, d\mu(x)+Q(z)\int_{-a}^{a}\sin zx\, d\mu(x)$. Then $F(z)$ is an entire function and $F(z)=0$ for $z=0, 1, 2, \ldots$. Let $K=a+(\pi-a)/2$. Then $a<K<\pi$. Also, $|P(z)|\leqq Me^{(\pi-a)|z|/2}$ and $|Q(z)|\leqq Me^{(\pi-a)|z|/2}$ for some M. Thus $|F(z)|\leqq M_1e^{K|z|}$ for some M_1. By Theorem A, $F(z)\equiv 0$.

$$P(z)\int_{-a}^{a}\cos zx\, d\mu(x)+Q(z)\int_{-a}^{a}\sin zx\, d\mu(x) = F(z) \equiv 0, \tag{2.1}$$

$$P(-z)\int_{-a}^{a}\cos zx\, d\mu(x)-Q(-z)\int_{-a}^{a}\sin zx\, d\mu(x) = F(-z) \equiv 0. \tag{2.2}$$

Multiplying (2.1) by $Q(-z)$ and (2.2) by $-Q(z)$ and adding, we get

$$[P(z)Q(-z)+P(-z)Q(z)]\int_{-a}^{a}\cos zx\, d\mu(x) \equiv 0$$

or

$$D(z)\int_{-a}^{a}\cos zx\, d\mu(x) \equiv 0.$$

Since $D(z)\not\equiv 0$, $\int_{-a}^{a}\cos zx\, d\mu(x)\equiv 0$. Similarly, multiplying (2.1) by $P(-z)$ and (2.2) by $-P(z)$ and adding, we get

$$D(z)\int_{-a}^{a}\sin zx\, d\mu(x) \equiv 0.$$

Thus $\int_{-a}^{a}\sin zx\, d\mu(x)\equiv 0$.

Since $\int_{-a}^{a}\cos nx\, d\mu(x)=\int_{-a}^{a}\sin nx\, d\mu(x)=0$ for $n=0, 1, 2, \ldots$ we have that all the Fourier coefficients of $d\mu(x)$ are zero. By the completeness of $\{\cos nx, \sin nx\}_{n=0}^{\infty}$ in $C[-a, a]$, we must have that $d\mu(x)\equiv 0$.

Theorem 14 proves that if $D(z)\not\equiv 0$, then $\{P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ is complete in $C[-a, a]$ for all a such that $0<a<\pi$. In a general sense, we have completeness in the "largest" interval under the "strongest" norm. The next theorem will prove that if $D(z)\equiv 0$, then we get incompleteness in $L^1[-\varepsilon, \varepsilon]$ for any $\varepsilon>0$, which is sort of the "smallest" interval under the "weakest" norm. Not only is the sequence incomplete, but even the addition of any finite number of integrable functions still leaves the sequence incomplete.

First we will prove the following lemma.

LEMMA 1. *Let $P(z)$ and $Q(z)$ be polynomials where $P(z)$ is even and $Q(z)$ is odd. Let $g_1(x), \dots, g_N(t)$ be any integrable functions on $[-a, a]$ where $0<a<\pi$. Then, there exists a continuous nontrivial function $f(x)$ on $[-a, a]$ such that*

$$P(z)\int_{-a}^{a} f(x)\cos zx\,dx + Q(z)\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0, \tag{1}$$

$$\int_{-a}^{a} f(x)g_n(x)\,dx = 0, \qquad n=1,\dots,N. \tag{2}$$

Proof. Assume $P(z)=a_0+a_2z^2+\cdots+a_{2K}z^{2K}$ and $Q(z)=a_1z+\cdots+a_{2K-1}z^{2K-1}$ where any of the a_i (including a_{2k}) may be zero. Let $M=N+K$. Let $g(x)$ be in $C^{2M}[-\pi,\pi]$, nontrivial, odd and zero on $[-\pi,-a]$ and $[a,\pi]$. Then $g(\pm a) = g'(\pm a)=\cdots=g^{(2M)}(\pm a)=0$.

Integration by parts, combined with the vanishing of $g(x)$ and its first $2M$ derivatives at $\pm a$, gives us that

$$\int_{-a}^{a} g^{(2n-1)}(x)\cos zx\,dx = (-1)^{n-1}z^{2n-1}\int_{-a}^{a} g(x)\sin zx\,dx, \qquad n\leqq M, \tag{2.3}$$

and

$$\int_{-a}^{a} g^{(2n)}(x)\sin zx\,dx = (-1)^{n}\,z^{2n}\int_{-a}^{a} g(x)\sin zx\,dx, \qquad n\leqq M. \tag{2.4}$$

By the fact that $g(x)$ and all its even numbered derivatives are odd and all its odd numbered derivatives are even, we have

$$\int_{-a}^{a} g^{(2n-1)}(x)\sin zx\,dx = 0, \qquad n\leqq M, \tag{2.5}$$

and

$$\int_{-a}^{a} g^{(2n)}(x)\cos zx\,dx = 0, \qquad n\leqq M. \tag{2.6}$$

Let $f(x)=c_0g(x)+c_1g'(x)+\cdots+c_{2M}g^{(2M)}(x)$ where $\{c_i\}_{i=0}^{2M}$ will be determined later.

Let $F(z)=P(z)\int_{-a}^{a}f(x)\cos zx\,dx+Q(z)\int_{-a}^{a}f(x)\sin zx\,dx$. By equations (2.3), (2.4), (2.5) and (2.6),

$$F(z) = P(z)(c_1z-c_3z^2+\cdots+(-1)^{M-1}c_{2M-1}z^{2M-1})\int_{-a}^{a} g(x)\sin zx\,dx$$

$$+\,Q(z)(c_0-c_2z^2+\cdots+(-1)^{M}c_{2M}z^{2M})\int_{-a}^{a} g(x)\sin zx\,dx.$$

Let

$$R(z) = P(z)(c_1z-c_3z^3+\cdots+(-1)^{M-1}c_{2M-1}z^{2M-1}) + Q(z)(c_0-c_2z^2+\cdots+(-1)^M c_{2M}z^{2M}).$$

Then $F(z)=R(z)\int_{-a}^{a} g(x)\sin zx\,dx$.

$R(z)$ is an odd polynomial of degree at most $2M+2K-1$. Its coefficients are linear combinations of $\{c_i\}_{i=0}^{2M}$. For $R(z)$ to be identically zero, all its coefficients would have to be zero. That gives us $M+K$ linear homogeneous equations in $\{c_i\}_{i=0}^{2M}$ since there are $M+K$ odd integers between 1 and $2M+2K-1$. If, in addition, we require that

$$\int_{-a}^{a} f(x)g_n(x)\,dx = 0, \qquad n = 1,\ldots,N,$$

we get N more linear homogeneous equations in $\{c_i\}_{i=0}^{2M}$. Altogether we get $M+K+N$ equations in $2M+1$ unknowns. However, since $K+N=M$, we have more unknowns than equations and can solve nontrivially. Thus we have produced a continuous function $f(x)$ on $[-a, a]$ such that

(1) $$P(z)\int_{-a}^{a} f(x)\cos zx\,dx+Q(z)\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0,$$

(2) $$\int_{-a}^{a} f(x)g_n(x)\,dx = 0, \qquad n = 1,\ldots,N.$$

All that is left to prove is that $f(x)\not\equiv 0$. If it were, then $g(x)$ would be a nontrivial solution to the differential equation

$$c_{2M}Y^{(2M)}(x)+c_{2M-1}Y^{(2M-1)}(x)+\cdots+c_1Y'(x)+c_0Y(x) \equiv 0$$

satisfying the boundary conditions

$$Y(a) = Y'(a) = \cdots = Y^{(2M)}(a) = 0.$$

However, as is well known, this can only be solved by the trivial function. Hence, $f(x)$ is nontrivial.

THEOREM 15. *Let $P(z)$ and $Q(z)$ be polynomials satisfying $D(z)=P(z)Q(-z)+P(-z)Q(z)\equiv 0$ and let a be such that $0<a<\pi$. Then $\{P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ is incomplete in $L^1[-a, a]$. Moreover, if $g_1(x), g_2(x),\ldots, g_N(x)$ are any functions in $L^1[-a, a]$, then $\{g_1(x),\ldots, g_N(x), P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ is incomplete in $L^1[-a, a]$.*

Proof. *Case* I. $P(z)$ is even and $Q(z)$ is odd. It is a trivial calculation that, in this case, $D(z)\equiv 0$.

Since $P(z)$ is even and $Q(z)$ is odd, Lemma 1 applies. Therefore, there exists a continuous, nontrivial function $f(x)$ such that

(1) $$P(z)\int_{-a}^{a} f(x)\cos zx\,dx+Q(z)\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0$$

and

(2) $$\int_{-a}^{a} f(x)g_n(x)\,dg = 0, \qquad n = 1,\ldots,N.$$

(1) certainly implies that

$$P(n)\int_{-a}^{a} f(x)\cos nx\,dx + Q(n)\int_{-a}^{a} f(x)\sin nx\,dx = 0, \qquad n = 0, 1, 2, \ldots.$$

Since $f(x)$ is continuous on $[-a, a]$, $f(x) \in L^{\infty}[-a, a]$. Hence

$$\{g_1(x), \ldots, g_N(x), P(n)\cos nx + Q(n)\sin nx\}_{n=0}^{\infty}$$

is incomplete in $L^1[-a, a]$.

Case II. $P(z)$ is not even. $z[P(z)-P(-z)]$ is an even polynomial, while $z[Q(z)+Q(-z)]$ is odd. By Lemma 1, there exists a continuous nontrivial function $f(x)$ such that

(1) $$z[P(z)-P(-z)]\int_{-a}^{a} f(x)\cos zx\,dx + z[Q(z)+Q(-z)]\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0,$$

(2) $$\int_{-a}^{a} f(x)g_n(x)\,dx = 0, \qquad n = 1, 2, \ldots, N.$$

Since $P(z)Q(-z)+P(-z)Q(z)\equiv 0$, we have $Q(-z)\equiv -P(-z)Q(z)/P(z)$. Substituting for $Q(-z)$ in (1), and multiplying through by $P(z)/z[P(z)-P(-z)]$, [which we can do since $P(z)$ is not even], we get

$$P(z)\int_{-a}^{a} f(x)\cos zx\,dx + Q(z)\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0.$$

As in Case I, we now have incompleteness.

Case III. $Q(z)$ is not odd.

We follow the same procedure as in Case II except that we substitute for $P(-z)$ and multiply by $Q(z)/z[Q(z)+Q(-z)]$.

In Theorem 14 we proved that if $D(z)\not\equiv 0$, then we have completeness in $C[-a, a]$ for any a such that $0<a<\pi$. A question which arises is "Do we need all the terms from $n=0, 1, \ldots$ or can some be eliminated without affecting completeness?" In the following theorem we will prove that actually an infinite number of terms can be omitted, provided the omitted set of integers is "sparse" among the set of positive integers.

THEOREM 16. *Let S be a set of nonnegative integers such that there exists an $\alpha<1$ such that*

$$\sum_{n\in S;\, n\neq 0} \frac{1}{n^{\alpha}} < \infty.$$

Let $\bar{S}$ be the complement of S in the set of nonnegative integers. Let $P(z)$ and $Q(z)$ be polynomials and let a be such that $0<a<\pi$. Then if $P(z)Q(-z)+P(-z)Q(z)\not\equiv 0$, $\{P(n)\cos nx + Q(n)\sin nx\}n \in \bar{S}$ is complete in $C[-a, a]$.

Proof. Assume there exists a measure $d\mu(x)$ such that

$$P(n)\int_{-a}^{a} \cos nx\,d\mu(x) + Q(n)\int_{-a}^{a} \sin nx\,d\mu(x) = 0, \qquad n \in \bar{S}.$$

Let

$$F(z) = P(z)\int_{-a}^{a} \cos zx\, d\mu(x)+Q(z)\int_{-a}^{a} \sin zx\, d\mu(x).$$

Let $\varepsilon=(\pi-a)/3$. Then $|F(z)|\leqq Me^{(a+\varepsilon)|z|}$ for some M. Let

$$W(z) = z\prod_{n\in S;\, n\neq 0}(1-z/n)e^{z/n}.$$

By a theorem [1, p. 19], the order of $W(z)$ is α. Therefore $|W(z)|\leqq M_1e^{\varepsilon|z|}$ for some M_1. Let $G(z)=F(z)W(z)$. $G(z)$ is an entire function, vanishes at all the nonnegative integers and satisfies $|G(z)|\leqq MM_1e^{(a+2\varepsilon)|z|}$. Since $0<a+2\varepsilon<\pi$, by Theorem A, $G(z)\equiv 0$. As $W(z)$ is obviously not identically zero, we must have that $F(z)\equiv 0$. Therefore

$$P(n)\int_{-a}^{a} \cos nx\, d\mu(x)+Q(n)\int_{-a}^{a} \sin nx\, d\mu(x)=0, \qquad n = 0, 1, 2, \ldots.$$

By Theorem 14 $\{P(n)\cos nx+Q(n)\sin nx\}_{n=0}^{\infty}$ is complete in $C[-a, a]$. Hence $d\mu(x)\equiv 0$ and $\{P(n)\cos nx+Q(n)\sin nx\}$, $n\in\bar{S}$, is complete in $C[-a, a]$.

It is interesting to take some particular $P(z)$ and $Q(z)$ and see how the change of interval from $[0, \pi]$ to $[-a, a]$ affects completeness.

I. Let $P(z)\equiv 1$ and $Q(z)\equiv 0$. We then have $\{\cos nx\}_{n=0}^{\infty}$ which is complete in $C[0, \pi]$ and incomplete in $L^1[-\varepsilon, \varepsilon]$.

II. Let $P(z)\equiv 1$ and $Q(z)\equiv\lambda$. We then have $\{\cos nx+\lambda\sin nx\}_{n=0}^{\infty}$ which, by [2], is complete in $C[0, \pi]$ and in $C[-a, a]$ for any $a<\pi$. The difference is, that when we discard the constant function, we have incompleteness in some $L^P[0, \pi]$ spaces as well as in $C[0, \pi]$, whereas we still have completeness in $C[-a, a]$.

III. Let $P(z)\equiv z$ and $Q(z)\equiv\lambda$, $\lambda\neq 2ki$, k a nonzero integer. Then we have $\{n\cos nx+\lambda\sin nx\}_{n=1}^{\infty}$ which is incomplete in both $L^1[0, \pi]$ and $L^1[-\varepsilon, \varepsilon]$. The difference is that if we add in the constant function we get completeness in $C[0, \pi]$ and still have incompleteness in $L^1[-\varepsilon, \varepsilon]$ for any $\varepsilon>0$.

IV. Let $P(z)\equiv\lambda$ and $Q(z)\equiv z$. Then we have $\{\lambda\cos nx+n\sin nx\}_{n=0}^{\infty}$ which is complete in $L^P[0, \pi]$ for any $P\geqq 1$ and is incomplete in $L^1[-\varepsilon, \varepsilon]$ for any $\varepsilon>0$.

Bibliography

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.

2. R. P. Feinerman and D. J. Newman, *Completeness of* sin $(nx+\alpha)$ *on* $[0, \pi]$, Michigan Math. J. (to appear).

3. E. Titchmarsh, *Theory of functions*, 2nd ed., Oxford Univ. Press, London, 1939.

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